Second-order differential equation with linear dependence in the solution

Find the general solution of the following differential equation, considering the given initial conditions

$$y''(x) + 2y'(x) - 2 - \sin(2x) = 0 \quad \text{with} \quad y(0) = 1 \text{ and } y'(0) = 2.$$

Solution

First, we solve the homogeneous part:

$$y''(x) + 2y'(x) = 0$$

We propose

$$y_H = e^{rx}$$

Then:

$$y_H' = e^{rx}r$$
$$y_H'' = e^{rx}r^2$$

Substituting:

$$e^{rx}r^2 + 2e^{rx}r = 0$$
$$r^2 + 2r = 0$$
$$r(r+2) = 0$$

The solutions are $r_1 = 0$ and $r_2 = -2$. The homogeneous solution is:

$$y_H = C_1e^{-2x} + C_2$$

Next, we solve the particular part:

$$y''(x) + 2y'(x) = 2 + \sin(2x)$$

First, we solve the trigonometric part by proposing $A\sin(2x) + B\cos(2x)$ for this part. Then:

$$y_p = A\sin(2x) + B\cos(2x)$$

$$y_p' = 2A\cos(2x) - 2B\sin(2x)$$
$$y_p'' = -4A\sin(2x) - 4B\cos(2x)$$

Substituting:

$$-4A\sin(2x) - 4B\cos(2x) + 4A\cos(2x) - 4B\sin(2x) = 2 + \sin(2x)$$
$$\sin(2x)(-4A - 4B) + \cos(2x)(-4B + 4A) = 2 + \sin(2x)$$

Thus,

$$-4A - 4B = 1$$

and

$$-4B + 4A = 0$$

Adding the equations:

$$-8B = 1$$
$$B = -\frac{1}{8}$$

And

$$-4A + \frac{1}{2} = 1$$

Solving for A:

$$-\frac{1}{2} = 4A$$

Thus,

$$A = -\frac{1}{8}$$

We propose the other part of the particular solution, noting that it must be linearly independent of the homogeneous solution, so we multiply the constant by x:

$$y_p = Cx$$

Differentiating:

$$y_p' = C$$

$$y_p'' = 0$$

In our differential equation:

$$2C = 2$$

Thus:

$$C = 1$$

Finally, we have the particular solution:

$$y_p = -\frac{\sin(2x)}{8} - \frac{\cos(2x)}{8} + x$$

The general solution is the homogeneous solution plus the particular solution:

$$y_g = y_H + y_p = C_1 e^{-2x} + C_2 - \frac{\sin(2x)}{8} - \frac{\cos(2x)}{8} + x$$

Using the initial conditions:

$$1 = C_1 + C_2 - \frac{1}{8}$$

$$y' = -2C_1 e^{-2x} - \frac{\cos(2x)}{4} + \frac{\sin(2x)}{4} + 1$$

$$2 = -2C_1 + 1 - \frac{1}{4}$$

Solving for C_1:

$$C_1 = \frac{5}{8}$$

And

$$C_2 = \frac{7}{4}$$

Thus:

$$y_g = -\frac{5}{8}e^{-2x} + \frac{7}{4} - \frac{\sin(2x)}{8} - \frac{\cos(2x)}{8} + x$$